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Rossby-wave instability in viscous discs

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ABSTRACT

The Rossby wave instability (RWI), which depends on the density bumps and extremum in the vortensities in the differentially rotating discs, plays an important role in the evolution of the protoplanetary discs. In this article, we investigate the effect of viscosity on the non-axisymmetric RWI in the self-gravitating accretion discs. For this purpose, we add the viscosity to the work of Lovelace and Hohlfield (2013). Consideration of viscosity complicates the problem so that we use the numerical method to investigate the stable and unstable modes. We consider three ranges of viscosities: high viscosity in the ranges $0.1 \leq \alpha \leq 0.4$, moderate viscosity in the ranges $0.01 \leq \alpha < 0.1$, and low viscosity in the ranges $\alpha < 0.01$. The results show that the occurrence of the RWI is related to the value of viscosity so that the effect of high viscosity is important, while the low viscosity is negligible. These results may be applied for the study of the RWI role in planet formation and angular momentum transport for different kinds of the protoplanetary discs with different viscosities.

Subject headings: accretion, accretion discs - instabilities - hydrodynamics - waves - spiral galaxies

1. Introduction

The theory of Rossby wave instability (RWI) was developed by Lovelace et al. (1999) and Li et al. (2000) for thin accretion discs with negligible viscosity and self-gravity (Lovelace & Hohlfield 2013). The criterion to obtain the RWI depends on the density bump and an extremum in the vortensity. A density bump could arise at the radial boundary of the dead zone (e.g., Tagger & Varniere 2006; Lyra et al. 2009; Ataiee et al. 2013) and vortensity is

defined by $\zeta = \kappa^2/(2\Omega\Sigma)$, where Σ is the disc surface density, Ω is the disc rotation rate and $\kappa^2 = 4\Omega^2 + 2r\Omega d\Omega/dr$ is the square of the radial epicyclic frequency (e.g., Narayan et al. 1987; Drazin & Reid 2004; Varniere & Tagger 2006; Yu & Lai 2013). For two-dimensional (vertically integrated) barotropic discs, the RWI relies on the existence of an extremum in the background fluid vortensity. Since Rossby waves propagate along the gradient of vortensity, the instability can be understood as arising from the interaction between two Rossby waves propagating on each side of the vortensity extremum (Yu & Lai 2013).

Recently, the RWI has been reviewed and developed by some astronomers. For example, Lovelace and Hohlfield (2013) analyzed the RWI in a continuum of thin disc models ranging from self-gravitating to non-selfgravitating cases. They found that the important quantities determining the stability/instability are: (1) the parameters of the bump (or depression) in the inverse potential vortensity (2) the Toomre Q parameter of the disc, and (3) the dimensionless azimuthal wavenumber of the perturbation $\bar{k}_\phi = mQh/r_0$, where r_0 represents the bump place, h is the half-thickness of the disc and $m = 1, 2, \dots$ is the azimuthal mode number. As another study, we can mention the work of Yu and Lai (2013) in which they studied the effect of large-scale magnetic fields on the non-axisymmetric RWI in the accretion discs. They show that the instability develops around a density bump, which is likely present in the transition region between the active and dead zones of the protoplanetary discs. Some of the simulations on the RWI have been performed with considering the viscosity (e.g., Varniere & Tagger 2006), and some of them have been done in the MHD resistive discs (e.g., Lyra & Mac Low 2012).

The viscosity has a significant impact on the evolution of the thin discs. The standard model of the viscous accretion discs was formulated in the well known papers of Shakura (1972) and Shakura and Sunyaev (1973). The viscosity affects the density bump, which is likely present in the transition region between the active zone and dead zone of the protoplanetary discs. The viscous torque at the transition has a component proportional to the negative of the viscosity gradient, so material is accelerated outward in the inner dead zone boundary (negative viscous gradient) and inward in the outer dead zone boundary (positive viscous gradient). This modifies the potential vortensity profile at these transitions, triggering the RWI (Lyra & Mac Low 2012). In this study, we investigate the effect of viscosity on the RWI. For this purpose, we follow the work of Lovelace and Hohlfield (2013) by adding the α -model viscosity to their work. Formulation of the problem is given in section 2. The results and astrophysical implications are given in section 3, and section 4 is devoted to conclusion and discussion.

2. Formulation of the Problem

We use cylindrical coordinate, (r, θ, z) , centered on the accreting object and make the following standard assumptions:

(1) The equilibrium has the flow velocity $\mathbf{u} = u_\phi \hat{e}_\phi = r\Omega(r)\hat{e}_\phi$ where $\Omega(r)$ is angular velocity at radius r . That is, the accretion velocity u_r and the vertical velocity u_z are assumed negligible compared with u_ϕ .

(2) The viscosity is $\nu = \alpha c_s H$ where c_s is the sound speed, H is the disc height, and α is an important free parameter between zero (no accretion) and approximately one. In other words, α is a parameter that is used to model the unknown sources of angular momentum transport (Pringle 1981).

(3) The gravitational potential Φ is given by $\nabla^2 \Phi = 4\pi G$, where G is the gravitational constant.

(4) The equilibrium flow satisfies $-\Sigma r \Omega^2 = -dP/dr - \Sigma \nabla \Phi$, where P is the vertically integrated pressure and Σ is the surface density.

The perturbed quantities are: density, $\tilde{\Sigma} = \Sigma + \delta\Sigma(r, \phi, t)$; pressure, $\tilde{P} = P + \delta P(r, \phi, t)$; flow velocity, $\tilde{\mathbf{u}} = \mathbf{u} + \delta\mathbf{u}(r, \phi, t)$ where $\delta\mathbf{u} = (\delta u_r, \delta u_\phi, 0)$. The equations for the perturbed flow are

$$\frac{D\tilde{\Sigma}}{Dt} + \tilde{\Sigma} \nabla \cdot \tilde{\mathbf{u}} = 0, \quad (1)$$

$$\frac{D\tilde{\mathbf{u}}}{Dt} = -\frac{\nabla \tilde{P}}{\tilde{\Sigma}} - \nabla \Phi + \nabla \cdot \Upsilon, \quad (2)$$

$$\frac{DS}{Dt} = 0, \quad (3)$$

where Υ is viscous stress tensor, $D/Dt = \partial/\partial t + \tilde{\mathbf{u}} \cdot \nabla$, and $S = \tilde{P}/(\tilde{\Sigma})^\gamma$ is the entropy of the disc matter. Since we consider radially localized modes in the sense that perturbation extends over a radial region Δr around r_0 with $(\Delta r)^2 \ll r_0^2$, we use perturbations as follows: $-f_0 \exp(ik_r r + im\phi - i\omega t)$, where amplitude f_0 is a constant, k_r is the radial wavenumber of the perturbation, $m = 0, 1, 2, \dots$ is the azimuthal mode number and $\omega = \omega_r + i\omega_i$ in which for growing modes of interest, $\omega_i > 0$. The perturbations are the first terms of Fourier expansion and we neglect other terms for simplification.

From equation (1), we have

$$\Delta\omega\delta\Sigma = \left(k_r\Sigma - i\frac{\partial\Sigma}{\partial r}\right)\delta u_r + k_\phi\Sigma\delta u_\phi, \quad (4)$$

where

$$\Delta\omega(r) = \omega - m\Omega(r). \quad (5)$$

From equation (2) we have

$$i\Delta\omega\delta u_r + 2\Omega\delta u_\phi = \frac{ik_r}{\Sigma}\delta P - \frac{\delta\Sigma}{\Sigma^2}\frac{dP}{dr} + ik_r\delta\Phi, \quad (6)$$

$$i\Delta\omega\delta u_\phi - \frac{\Omega_r^2}{2\Omega}\delta u_r = ik_\phi\frac{\delta P}{\Sigma} + ik_\phi\delta\Phi - \nu\left(\frac{ik_r}{r}\delta u_\phi - \frac{\delta u_\phi}{r^2} - k_r^2\delta u_\phi + \frac{2ik_\phi}{r}\delta u_r - k_\phi^2\delta u_\phi\right). \quad (7)$$

Here, $\Omega_r = [r^{-3}d(r^4\Omega^2)/dr]^{\frac{1}{2}}$ is the radial epicyclic frequency, and $k_\phi = m/r$ is the azimuthal wavenumber. For an approximately Keplerian disc, $\Omega_r \approx \Omega$. From equation (3) and the definition of entropy, we have

$$\delta P = c_s^2\delta\Sigma - \frac{i\Sigma c_s^2}{\Delta\omega L_s}\delta u_r, \quad (8)$$

where, $c_s = (dP/d\Sigma)^{1/2}$ is the effective sound speed and $L_s^{-1} = \gamma^{-1}d\ln(S)/dr$ with L_s the length-scale of the entropy variation in the disc. To simplify the subsequent calculations, we consider the homentropic case where $L_s \rightarrow \infty$ (Lovelace & Hohlfield 2013). For an approximately Keplerian disc with $\Omega_r \approx \Omega$, we can neglect of dP/dr in equation (6). In this case, we can rewrite equations (6) and (7) as follows

$$i\Delta\omega\delta u_r + 2\Omega\delta u_\phi = ik_r\delta\Psi, \quad (9)$$

$$A_0\delta u_\phi - A_1\delta u_r = ik_\phi\delta\Psi, \quad (10)$$

where

$$A_0 = \left(i\Delta\omega + \frac{i\nu k_r}{r} - \frac{\nu}{r^2} - \nu(k_r^2 + k_\phi^2)\right), \quad A_1 = \left(\frac{\Omega_r^2}{2\Omega} - \frac{2i\nu k_\phi}{r}\right), \quad \delta\Psi = c_s^2\frac{\delta\Sigma}{\Sigma} + \delta\Phi. \quad (11)$$

Equations (9) and (10) can be solved to give

$$\delta u_r = \frac{\delta\Psi}{2\Omega A_1 + iA_0\Delta\omega} (iA_0k_r - 2ik_\phi\Omega), \quad (12)$$

$$\delta u_\phi = \frac{\delta\Psi}{2\Omega A_1 + iA_0\Delta\omega} (iA_1k_r - k_\phi\Delta\omega). \quad (13)$$

By substituting equations (12) and (13) into (4) we obtain

$$(2\Omega A_1 + iA_0\Delta\omega)\Delta\omega\frac{\delta\Sigma}{\Sigma} = i\left(k_r - \frac{i}{\Sigma}\frac{\partial\Sigma}{\partial r}\right)(A_0k_r - 2k_\phi\Omega)\delta\Psi + i(A_1k_\phi k_r + ik_\phi^2\Delta\omega)\delta\Psi. \quad (14)$$

The perturbation of the gravitational potential is give by

$$\nabla^2\Phi = 4\pi G\delta\Sigma\delta(z). \quad (15)$$

The WKBJ solution of equation (15) gives $\delta\Phi = -2\pi G\Sigma/|\mathbf{k}|^2$ where $\mathbf{k} = k_r\hat{e}_r + k_\phi\hat{e}_\phi$ (Lovelace & Hohlfield 2013). The WKBJ approximation allows performing a local perturbation stability analysis of the perturbed disc (no need to exact boundary conditions because the analysis is local and not global). The main simplification introduced by the WKBJ approximation mathematically is that the Poisson equation for the perturbation becomes a (local) algebraic relation between the surface density of the spiral wave and its potential. Therefore, for $\delta\Psi$, we can write

$$\delta\Psi = \left(1 - \frac{2k_c^2}{|\mathbf{k}|^2}\right)c_s^2\frac{\delta\Sigma}{\Sigma}, \quad (16)$$

where

$$k_c = \left(\frac{\pi G\Sigma}{c_s^2}\right)^{\frac{1}{2}}, \quad (17)$$

is a characteristic wavenumber. From equation (14) and (16), we can write dispersion relation as follows

$$(2\Omega A_1 + iA_0\Delta\omega)\Delta\omega = \left(c_s^2 - \frac{2k_c^2c_s^2}{|\mathbf{k}|^2}\right) \left[\left(k_r - \frac{i}{\Sigma}\frac{\partial\Sigma}{\partial r}\right) (iA_0k_r - 2ik_\phi\Omega) + (iA_1k_\phi k_r - k_\phi^2\Delta\omega) \right]. \quad (18)$$

For obtaining $\partial\Sigma/\partial r$, we consider a density bump in a thin accretion disc of the form (see Lovelace et al. 1999)

$$\frac{\Sigma(r)}{\Sigma_0} = 1 + (F - 1) \exp\left[-\frac{(r - r_0)^2}{2\Delta^2}\right], \quad (19)$$

where the subscript '0' implies that the quantities evaluated at $r = r_0$ and Σ_0 , F and Δ are constants, respectively. Following Yu & Lai (2013), we chose $0.4 < r/r_0 < 1.6$, $F = 1.2$ and $\Delta = 0.05r_0$ through this paper. We further assume $c_s = 0.1r_0\Omega_0 = \text{constant}$.

For axisymmetric perturbations ($k_\phi = 0 = m$) of smooth disc without viscosity ($\nu = 0$) and with neglecting the radial variation of Σ , we can write (Safranov 1960; Toomre 1964)

$$\omega^2 = \Omega_r^2 + k_r^2c_s^2 - 2k_c^2c_s^2. \quad (20)$$

The minimum of $[\omega(k_r)]^2$ occurs at $k_r = k_c$ where $\omega^2 = \Omega^2 - k_c^2c_s^2$. Therefore, if we define $k_{c*} = \Omega_r/c_s$, then for $k_c < k_{c*}$, the minimum of ω^2 is positive and the axisymmetric perturbations

are stable. Conversely for $k_c > k_{c*}$, the perturbations are unstable. With (Lovelace & Hohlfield 2013)

$$Q = \frac{k_{c*}}{k_c} = \frac{\Omega_r c_s}{\pi G \Sigma}, \quad (21)$$

the axisymmetric perturbations are stable (unstable) for $Q > 1$ ($Q < 1$) (Toomre 1964). The minimum value of the squared frequency is $\omega(kr)^2 = (k_{c*} c_s)^2 (1 - Q^{-2})$.

To obtain non-axisymmetric modes, by choosing following dimensionless parameters and variables

$$\begin{aligned} R = \frac{r}{r_0}, \quad \varpi = \frac{r_0 \Delta \omega}{c_s}, \quad \gamma = \frac{r_0 \Omega}{c_s}, \quad \gamma_r = \frac{r_0 \Omega_r}{c_s}, \quad \varrho = \frac{\Sigma}{\Sigma_0}, \quad \mu = \frac{\nu}{r_0 c_s}, \\ \sigma_r = k_r r_0, \quad \sigma_\phi = k_\phi r_0, \quad \sigma_c = k_c r_0, \end{aligned} \quad (22)$$

we can rewrite equation (18) as follows

$$\varpi^3 + C_2 \varpi^2 + C_1 \varpi + C_0 = 0, \quad (23)$$

where C_2 , C_1 and C_0 are complex coefficients in the equations below

$$C_2 = \left(\frac{\mu \sigma_r}{R} + \frac{i \mu}{R^2} + i \mu (\sigma_r^2 + \sigma_\phi^2) \right), \quad (24)$$

$$C_1 = - \left(1 - \frac{2\sigma_c^2}{\sigma_r^2 + \sigma_\phi^2} \right) \left[\sigma_\phi^2 + \left(\sigma_r - \frac{i}{\varrho} \frac{\partial \varrho}{\partial R} \right) \sigma_r \right] - \left(\gamma_r^2 - \frac{4i \gamma \mu \sigma_\phi}{R} \right), \quad (25)$$

$$\begin{aligned} C_0 = & \left(1 - \frac{2\sigma_c^2}{\sigma_r^2 + \sigma_\phi^2} \right) \\ & \left[\left(\frac{i}{\varrho} \frac{\partial \varrho}{\partial r} - \sigma_r \right) \left(\left(\frac{\mu \sigma_r}{R} + \frac{i \mu}{R^2} + i \mu (\sigma_r^2 + \sigma_\phi^2) \right) \sigma_r + 2i \sigma_\phi \gamma \right) + i \left(\frac{\gamma_r^2}{2\gamma} - \frac{2i \mu \sigma_\phi}{R} \right) \sigma_\phi \sigma_r \right]. \end{aligned} \quad (26)$$

The characteristic equation (23) allows determination of ϖ , and must be solved numerically to determine stable and unstable modes. For this purpose, we use the Laguerre method (Press et al. 1992) to obtain its three roots.

3. Results and Astrophysical Implications

In this section, we consider the results and their astrophysical implications with an emphasis on the formation of planets through accretion discs. We investigate the problem with three ranges of viscosities as follows: high viscosity in the ranges $0.1 \leq \alpha \leq 0.4$, moderate viscosity in the ranges $0.01 \leq \alpha < 0.1$, and low viscosity in the ranges $\alpha < 0.01$. In each range, we choose an arbitrary value of 0.2, 0.08 and 0.006 to represent the high, moderate and low viscosity, respectively. The stable and unstable regions versus to the azimuthal mode number, m , and the non-dimensional radial wavenumber, σ_r , are shown in Fig.(1) in which the unstable regions are depicted by dark lines. The Fig.(1) shows the effects of viscosity on the stable and unstable modes so that the perturbations corresponding to small azimuthal mode numbers may be damped for some non-dimensional radial wavenumbers. Also, Fig.(2) shows the growth rate of unstable modes versus to the non-dimensional radial wavenumber.

Fig.(1)-(a) and Fig.(2)-(a) are assigned to low viscosities which occur in weakly ionized protoplanetary discs around T-Tauri stars. In this case, the viscosity can be removed from the RWI problem because it does not have important impact. In other words, the low viscosity is not able to damp the perturbations of the Rossby waves.

Fig.(1)-(b) and Fig.(2)-(b) are assigned to moderate viscosities. The results of these figures can be utilized in the study of the RWI in the partially ionized protoplanetary discs around the stars of a young binary system (e.g., Hartmann 2007). According to these figures, we can deduce that the RWI, for some radial wavenumbers and azimuthal mode numbers ($m < 4$), may play an important role in the planetesimal formation and angular momentum transport through these protoplanetary discs.

Fig.(1)-(c) and Fig.(2)-(c) are assigned to high viscosities. Fig.(1)-(c) shows that the stable regions decrease with increasing the azimuthal mode number, m , and totally disappear for $m > 7$. These results can be utilized in the study of the RWI in fully-ionized protoplanetary discs, dwarf-nova accretion discs and hot discs around supermassive black holes (e.g., King et al. 2007). The temperature of hot disc increases the viscosity (e.g., Gholipour & Nejad-Asghar 2013) which is able to damp the perturbation of the Rossby waves (corresponding to small azimuthal mode number for $m < 7$). Also, these figures show that the RWI doesn't have important role in planetesimal formation and angular momentum transport in fully-ionized protoplanetary discs, dwarf-nova accretion discs and hot discs around supermassive black holes.

The growth rate versus to the viscosity coefficient is shown in Fig.(3). It implies that we have larger growth rate for smaller viscosity. The growth rate of each plot is found to be not only dependent on the viscosity coefficient but also dependent on the azimuthal

mode numbers. The growth rate and the area under the curve increase with decreasing the viscosity coefficient and with increasing of m . Obviously, Fig.(3) shows that for $m = 1$ and $m = 2$, the instability doesn't occur for high viscosity unless in small radial wavenumbers, and the distance between the lines becomes smaller with decreasing wavenumber. This figure has important results in obtaining timescale of planetesimal formation by the RWI in the protoplanetary discs. The rate of planetesimal formation is faster for a protoplanetary disc with low viscosity than one with high viscosity. This subject may be considered as a responsible process in problem of disc dispersal (Bodenheimer 2011).

The non-dimensional critical wavelength ($\tilde{\lambda}_{crit} = \lambda_{crit}/r_0$) versus the viscosity coefficient is shown in Fig.(4). It explains what would be the viscosity necessary to prevent the RWI from growing in a thin accretion disc. Since the long and intermediate wavelengths ($\lambda > r_0$) are not important in planetesimal formation process, the maximum of non-dimensional critical wavelength is considered equal to one. For example, if we assume $\alpha > 0.1$ for a hot disc (such as supermassive black hole accretion disc), the planetesimal formation by the RWI has less chance for occurrence in $m = 1$ than $m = 2$ or $m = 3$. Thus, we can neglect the contribution of $m = 1$ for a hot disc around supermassive black hole with $\alpha > 0.22$ (see Fig.(4)). In this case, the larger numbers of the azimuthal mode number may be contributed.

4. Conclusion and Discussion

In this paper, we have carried out linear analysis of the RWI in accretion discs including the viscosity. The RWI may play an important role in planetesimal formation and angular momentum transport in weakly ionized protoplanetary discs. In the study of Lovelace and Hohlfield (2013), the viscosity was ignored and the self-gravity was emphasized, but we added the viscosity in their self gravitating work. The viscosity complicates the problem, thus we used the numerical methods to investigate the stable and unstable modes. The results show that the occurrence of the RWI is related to the value of viscosity. Since the hot discs have higher viscosity than other discs (e.g., Gholipour & Nejad-Asghar 2013; King et al. 2007), the high viscosity is able to damp the perturbations corresponding to small azimuthal mode numbers. Our results indicate that the RWI may play more important role in weakly ionized thin disc than the fully or partially ionized one. In other words, the contribution of low viscosity may be ignored in the RWI. Here, we also considered the results of King et al. (2007) which considered observational and theoretical estimates of the accretion disc viscosity coefficient. They found that in thin, fully ionized discs, the best observational evidence suggests a typical range $\alpha \sim 0.1 - 0.4$, whereas the relevant numerical simulations tend to derive estimates for α which are an order of magnitude smaller. To compare with

simulation works, we can mention the work of Varniere and Tagger (2006). They found the consequences on the disc dynamics of the presence of dead zone, where the transport of matter and angular momentum (with a turbulent viscosity) is significantly lower than elsewhere in the disc. Our results are in agreement with this simulation work.

The results show the RWI can be damped in the hot discs so that it doesn't have an important role in the planetesimal formation in these discs. In this case, other processes can be considered according to the high viscosity. For example, Gholipour and Nejad-Asghar (2013) introduced viscothermal instability that the viscosity of hot disc causes thermal instability which may lead to planetesimal formation through discs. Also, the results of this paper can be considered in problem of disc dispersal. Numerous mechanisms, most still under investigation, have been proposed to explain the fact that observational evidence for the presence of discs around newly formed stars disappears once the stars reach an age of 1-10 Myr (Bodenheimer 2011). Our results predict that the rate of planetesimal formation is faster for a protoplanetary disc with low viscosity than a protoplanetary disc with high viscosity (see Fig.(3)). Since the discs around newly formed stars have low viscosity, $\alpha < 0.01$ (Bodenheimer 2011), we expect the disc disperses faster around protostar than around super-massive black holes. Briefly, the results show that the important quantities determining the stability/instability are: (1) the viscosity at region of the density bump (2) the azimuthal mode number (3) the radial wavelength. These results may explain why the formation of planets is impossible in some regions.

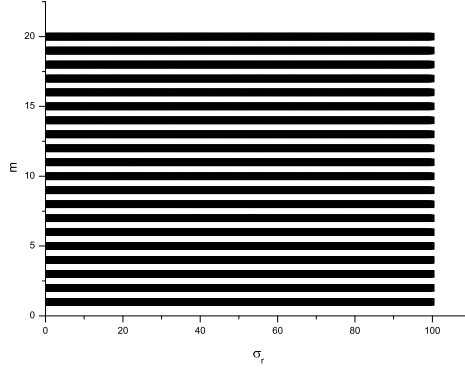
Acknowledgments

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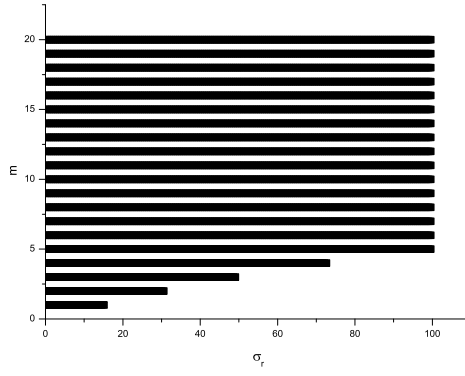
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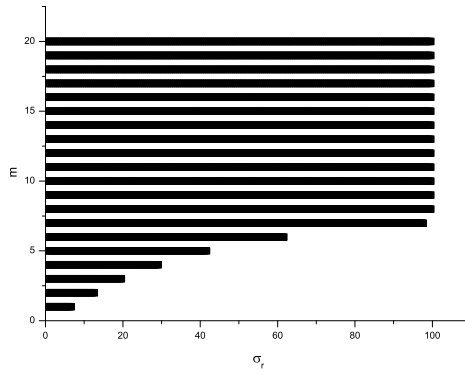
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(a)

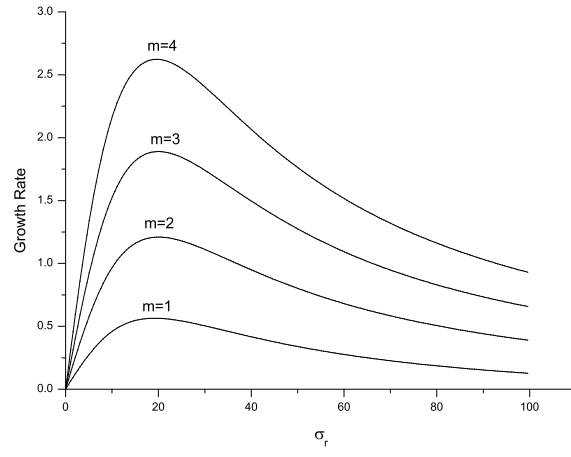


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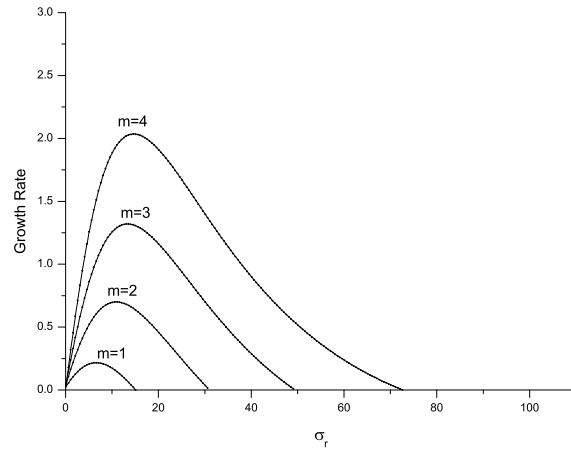


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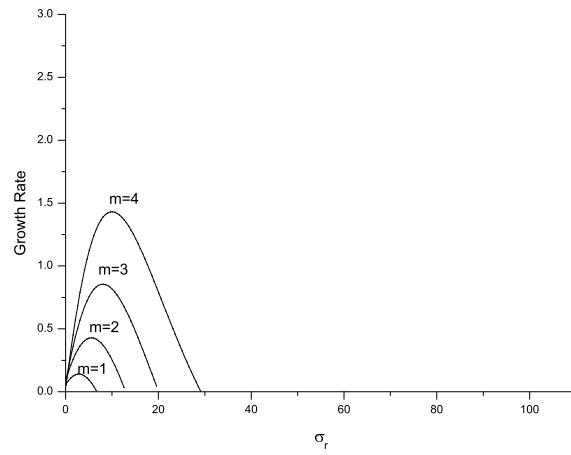
Fig. 1.— The stable and unstable regions versus to the azimuthal mode number, m , and the non-dimensional radial wavenumber, σ_r , are shown in Fig.(1) in which the unstable regions are depicted by dark lines for cases with (a) $\alpha = 0.006$, (b) $\alpha = 0.08$ and (c) $\alpha = 0.2$.



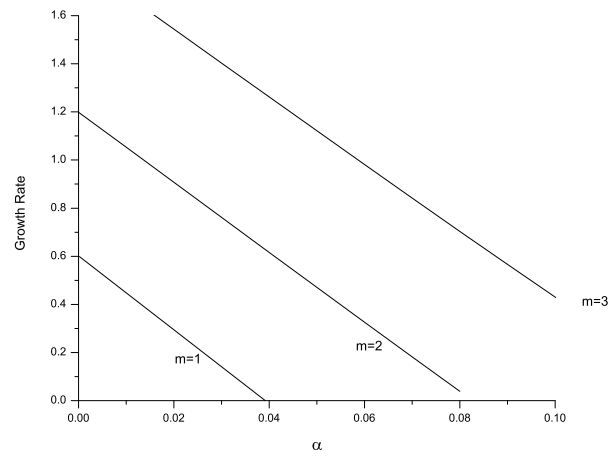
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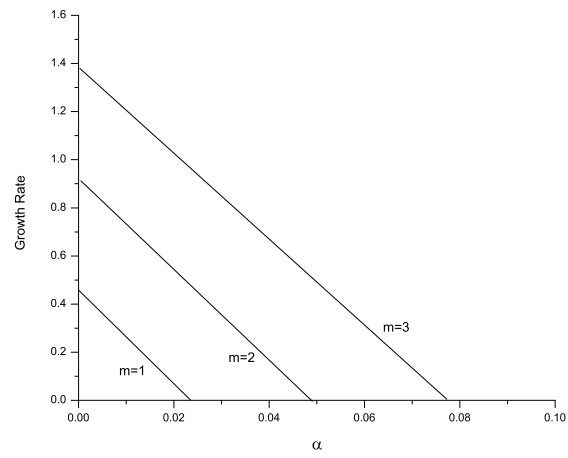
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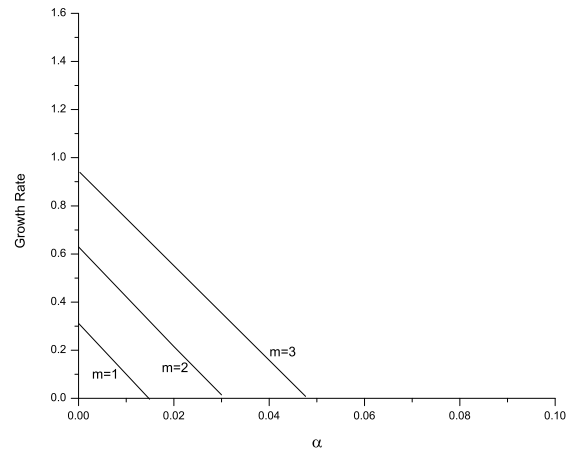
(c)



(a)



(b)



(c)

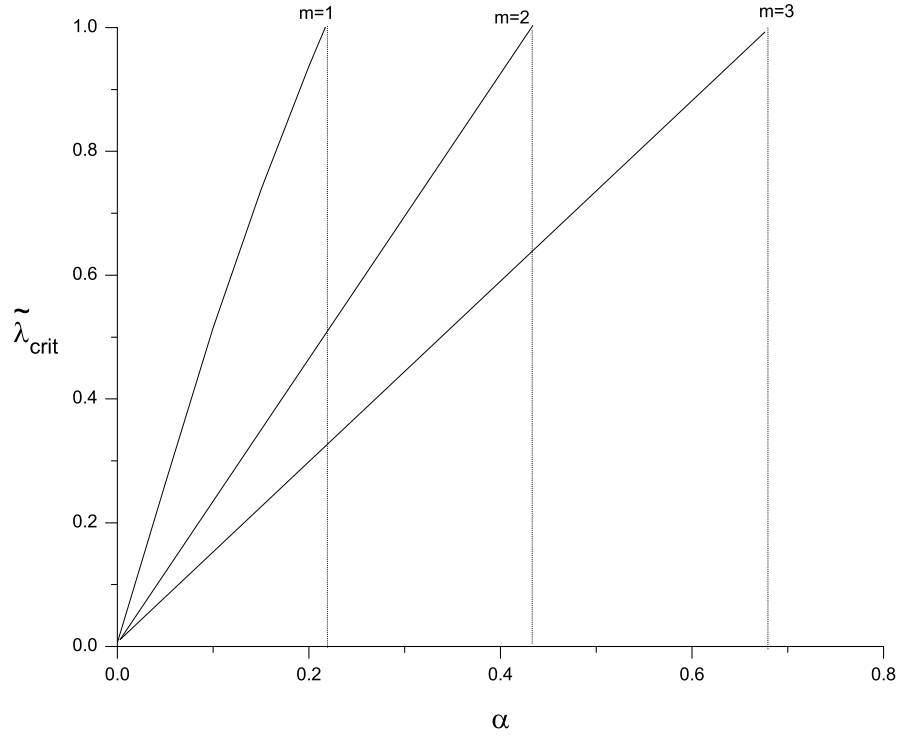


Fig. 4.— The non-dimensional critical wavelength ($\tilde{\lambda}_{crit} = \lambda_{crit}/r_0$) versus to α (the viscosity coefficient) with $m = 3, 2, 1$.